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Symbolic Computation to Estimate Two-Sided Boundary Conditions in Two-Dimensional Conduction Problems

Ching-yu Yang*
National Kaohsiung Institute of Technology, Kaohsiung City, Taiwan 807, Republic of China

A method using symbolic computation is proposed to determine the boundary conditions in two-dimensional inverse heat conduction problems. The method uses symbols to represent the unknown boundaries and then executes the finite difference method to calculate the temperature field. The calculated results are expressed explicitly as functions of the undetermined boundaries. Then, a direct comparison between the output symbolic temperature and the measurement temperature are made to construct a set of linear equations. Thus, the linear least-squares method is adopted to solve the linear equations. Finally, the unknown boundary conditions are determined. Results from the examples confirm that the proposed method is applicable in solving the multidimensional inverse heat conduction problems. In the example problems, three kinds of measuring methods are adopted to estimate the surface temperature. The result shows that only three measuring points are needed to estimate the surface temperature when measurement errors are neglected. When measurement errors are considered, seven measuring points are required to increase the congruence of the estimated results to the exact solutions.

Nomenclature

a, b	= undetermined coefficient
$f_1(y, t)$, $f_2(x, t)$	= estimated boundary conditions
$\bar{m}$, $\bar{n}$	= upper bound of the indices of coefficients
T	= temperature
t	= temporal coordinate
x, y	= spatial coordinate
Δt	= increment of temporal domain
Δx, Δy	= increment of spatial coordinate
$\boldsymbol{\lambda}$	= random number
$\phi_m(y, t)$, $\varsigma_n(x, t)$	= basis function
ω	= bound of random number

Subscripts

exact	= exact temperature
i, j, k	= indices
m, n	= indices for undetermined coefficients
measurement	= measurement temperature

Superscripts

i, j, k	= indices
$\bar{}$	= dimensional parameters
$\hat{}$	= exact boundary condition

Introduction

THE potential of the symbolic computation to improve the analysis of engineering problems has been recognized. The applications of symbolic computation in the field of analysis include solid mechanics, heat transfer, supersonic transition, gas kinetic, fluid, dynamics, differential geometry, and control systems.[1–4] However, there are only a limited number of researches that adopted symbolic computation into the field of inverse problems.[5]

The inverse heat conduction problems deal with the determination of the crucial parameters in analysis such as the heat transfer coefficient, contact conductance, surface heat flux, internal energy source, and thermal properties, etc. They have been widely applied in many design and manufacturing problems, especially when the direct measurements of the surface conditions are not possible. For example, it would be difficult to measure the temperature or heat flux at the tool–work interface of machine cutting, or at the inside of a combustion chamber, or at the outer surface of a re-entry vehicle. Wide attention has been called to the inverse problem, and most studies employ the numerical methods[6–13] to determine the unknown conditions in the inverse problems. In the numerical methods, the inverse problem is formulated from the finite difference, the finite element, or the boundary element methods to calculate the responses of the system. To determine unknown boundary conditions, these methods have often been combined with the optimization algorithms such as regularization technique, the sequential regularization approach and, the adjoint equation approach coupled to the conjugate gradient method, Newton–Raphson, genetic algorithms, and the multidimensional simplex method.

Based on the numerical approach, the determination of the boundary conditions includes two phases: 1) the process of the analysis and 2) the process of the optimization. In the analysis process, the boundary condition is assumed and then the temperature field of the differential governing equation is solved through the numerical method. Solutions from the previous process are integrated with the temperature measured at the interior point of the solid. Thus, a nonlinear least-squares problem is formatted for the process of optimization. In the optimization process, an optimizer is used to guide the exploring points systematically to approach for the boundary conditions. Then the newly guessed boundary conditions are substituted for the unknown boundary conditions in the following analysis process. As such, several iterations to solve the governing equation and nonlinear optimization problem are needed for obtaining the undetermined boundary conditions. Additionally, most inverse problems have only been used to deal with one-dimensional problems, but few researchers have solved two- or three-dimensional problems, because those problems are more sophisticated.

The purpose of this research is to demonstrate the use of symbolic computation in an inverse problem to solve two-dimensional heat conduction problems. In the process of the symbolic approach, the boundary conditions are represented as the series form first, and the boundary conditions can be as-

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*Associate Professor, Department of Mold and Die Engineering.

sumed as the series form with linear combination of the unknown coefficients. Then, the direct symbolic finite difference analysis is performed to find the temperature field in the solid. The calculated results of temperature distribution are expressed as symbols, where the symbols describe the boundary conditions. Because the calculated temperature field is explicitly given as functions of the boundary condition, the inverse analysis can be directly achieved by the resulting symbolic expressions and measurement datum. As a result, the inverse problems become a set of equations with a linear combination of the unknown coefficients, which lead to a solution of the inverse problem through the linear least-squares error method. The unique feature of this approach is that the inverse computation is in a linear domain and the nonlinear optimization process used in the traditional approach can be eliminated.

This article includes four sections. In the first section, the current applications of the symbolic computation in science and engineering are introduced and the features of using symbolic computation in the inverse conduction problems are also stated. In the second section, the characteristics of a two-dimensional problem are delineated and an implicit alternating-direction finite difference method is presented. Meanwhile, the process of the proposed method is illustrated. Three examples with three-, five-, and seven-point measurements are employed to demonstrate and discuss the results of the proposed method in the third section. In the final section, the overall contribution and possible applications of this research to the field of inverse heat conduction problem are discussed.

Two-Dimensional Heat Conduction Problem Based on the Proposed Method

Problem Statement

Consider an infinitely long bar with constant thermal properties and a square cross section (see Fig. 1). The adiabatic conditions are applied at the side of $\bar{x} = 0$ and $\bar{y} = \bar{L}$. It is initially at a uniform temperature $\bar{T}_0$, and then suddenly a temperature function $\bar{f}_1(\bar{y}, \bar{t})$ is applied to the side $\bar{x} = \bar{L}$ and $\bar{f}_2(\bar{x}, \bar{t})$ to the side $\bar{y} = 0$. A dimensionless mathematical formation of the two-dimensional heat conduction problem is presented as follows:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\partial T}{\partial t} \qquad 0 < x < 1, \quad 0 < y < 1, \quad t > 0 \tag{1}$$

$$T(x, y, 0) = T_0 = 1 \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad t = 0 \tag{2}$$

$$T(1, y, t) = f_1(y, t) \quad x = 1, \quad 0 \leq y \leq 1, \quad t > 0 \tag{3}$$

$$T(x, 0, t) = f_2(x, t) \quad 0 \leq x < 1, \quad y = 0, \quad t > 0 \tag{4}$$

$$\frac{\partial T(x, t, t)}{\partial x} = 0 \qquad x = 0, \qquad 0 \leq y < 1, \qquad t > 0 \tag{5}$$

$$\frac{\partial T(x, y, t)}{\partial y} = 0 \qquad 0 \leq x < 1, \qquad y = 1, \qquad t > 0 \tag{6}$$

where the following dimensionless quantities are defined as

$$x = \bar{x}/L, \quad y = \bar{y}/L, \quad T = \bar{T}/\bar{T}_0, \quad k = \bar{k}/\bar{k}_r, \quad t = (\bar{k}_r/\bar{\rho}\bar{C}_p)(\bar{t}/\bar{L}^2)$$

$\bar{\rho}\bar{C}_p$ is the heat capacity per unit volume, $\bar{T}_0$ and $\bar{k}_r$ refer to the nonzero reference temperature and thermal conductivity, respectively. We assume $\bar{k} = \bar{k}_r$, $f_1(y, t) = \bar{f}_1(y, t)/\bar{T}_0$, and $f_2(x, t)/\bar{T}_0$ in the problem.

The inverse problem is to identify the applied unknown boundaries $f_1(y, t)$ and $f_2(x, t)$, from the temperature measurements taken at the interior points of the bar. The proposed method is to formulate a set of equations from the calculated symbolic temperature and the measured temperature. To find

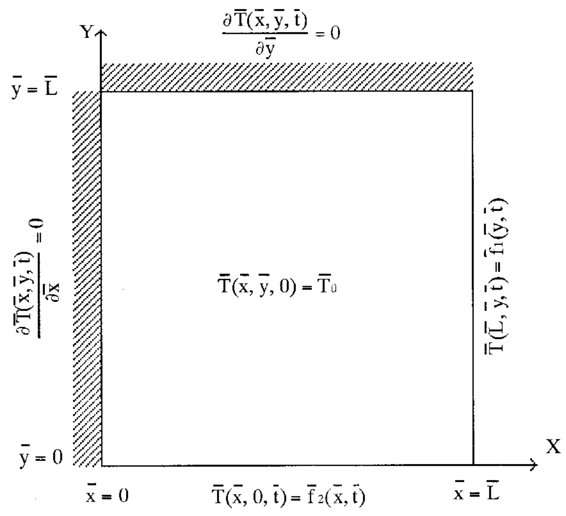

Fig. 1 Square cross section of an infinitely long bar.

the symbolic temperature, an implicit alternating-direction finite difference method is adopted. It is further described in the following paragraph.

Implicit Alternating-Direction Finite Difference Method with Symbolic Boundary Input

Suppose that the applied boundary conditions $f_1(y, t)$ and $f_2(x, t)$ are represented as the following linear forms with respect to a_m and b_n forms in a certain spatial and temporal domain:

$$f_1(y, t) = \sum_{m=0}^{\bar{m}} a_m \phi_m(y, t) \tag{7}$$

$$f_2(x, t) = \sum_{n=0}^{\bar{n}} b_n \varsigma_n(x, t) \tag{8}$$

where $\phi_m(y, t)$ and $\varsigma_n(x, t)$ can be any nonsingular function in the problem domain. a_m and b_n are the undetermined coefficients. $\bar{m}$ and $\bar{n}$ are positive integers.

After the boundary conditions are represented as the previous forms, an implicit alternating-direction finite difference method[14] is used to execute the analysis process. This algorithm uses a system of equations with a tridiagonal coefficient matrix, which affords a straightforward solution. The method employs two difference equations that are used in turn over successive time-steps, each a length of $\Delta t/2$. In the first equation, only the x direction is implicit and in the second equation only the y direction is implicit. Let $T^*_{i,j}$ be the intermediate value at the end of the first time-step $\Delta t/2$, we have

$$\frac{T^*_{i,j} - T_{i,j,k}}{\Delta t/2} = \frac{1}{\Delta x^2}(T^*_{i-1,j} - 2T^*_{i,j} + T^*_{i+1,j}) + \frac{1}{\Delta y^2}(T_{i,j-1,k} - 2T_{i,j,k} + T_{i,j+1,k}) \tag{9}$$

followed by

$$\frac{T_{i,j,k+1} - T^*_{i,j}}{\Delta t/2} = \frac{1}{\Delta x^2}(T^*_{i,j} - 2T^*_{i,j} + T^*_{i+1,j}) + \frac{1}{\Delta y^2}(T_{i,j-1,k+1} - 2T_{i,j,k+1} + T_{i,j+1,k+1}) \tag{10}$$

where Δx and Δy are the increment of spatial coordinate and Δt is the increment of temporal domain, i is the ith grid along

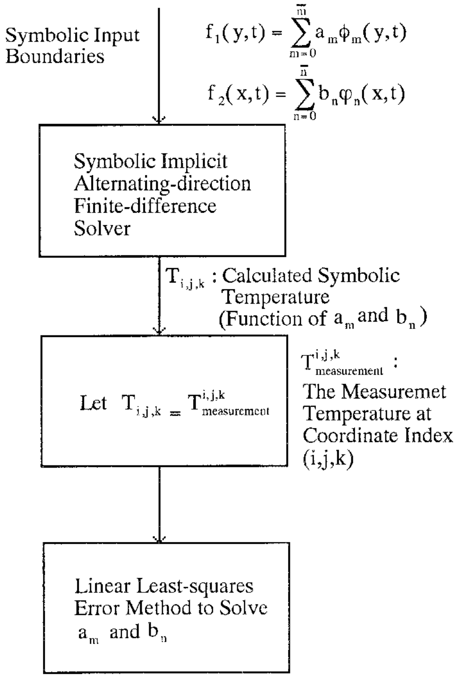

Fig. 2 Process of the proposed symbolic approach to determine boundary conditions in two-dimensional heat conduction problem.

with x coordinate, j is the jth grid along with y coordinate, k is the kth grid along with temporal-coordinate, and $T_{i,j,k}$ is the temperature at the grid point (i, j, k).

The intermediate value $T^{*}_{i,j}$ is solved in Eq. (9) and then it is used in Eq. (10). Thus it leads to the solution $T_{i,j,k+1}$ at the end of the whole time interval Δt. The stability of this procedure is proved to be unconditionally stable in Ref. 14.

After Eqs. (3) and (4) are substituted by Eqs. (7) and (8), and Eqs. (1–6) are discretized through the implicit alternating-direction finite difference method, the symbolic temperature can be solved directly.

Method to Determine Boundary Conditions

The method to determine boundary conditions can be best illustrated by Fig. 2. In this process, the input boundary conditions are given symbolically, i.e.,

$$\sum_{m=0}^{\bar{m}} a_m \boldsymbol{\phi}_m(y, t)$$

for $f_1(y, t)$ and

$$\sum_{n=0}^{\bar{n}} b_n \mathrm{s}_n(x, t)$$

for $f_2(x, t)$. Then, the symbolic temperature $T_{i,j,k}$ (function of a_m and b_n) is obtained by the symbolic solver, which is described in the last paragraph. After $T_{i,j,k}$ is calculated, the following step is to build a set of linear equations in which the calculated temperature is equal to the measured temperature ($T_{i,j,k} = T^{i,j,k}_{\text{measurement}}$). Thus, this set of equations can be solved through linear least-squares error method.[15] A special feature about the proposed process is that the inverse computation is in a linear domain and the nonlinear optimization process used in the traditional approach can be eliminated.

Results and Discussion

A two-dimensional inverse heat conduction problem with two unknown boundary conditions is used to demonstrate the strength of the proposed method, in which three-, five-, or seven-point measurements are taken independently in three examples to estimate the boundaries of the problem. The exact temperature and boundary conditions used in the following examples are preselected so that these functions can satisfy Eqs. (1–6). The accuracy of the proposed method is assessed by comparing the estimated boundary conditions with the preselected boundary conditions. Meanwhile, the simulated temperature measurement is generated from the preselected exact temperature in each problem and it is presumed to have measurement errors. In other words, the random errors of measurement are added to the exact temperature. It can be shown in the following equation:

$$T_{\text{measurement}} = T_{\text{exact}} + \boldsymbol{\lambda} T_{\text{exact}} \tag{11}$$

$$\lambda \leq |\omega| \tag{12}$$

where λ is the random error of measurement, and ω is the bound of λ. T_{exact} in Eq. (11) is the exact temperature and, $T_{\text{measurement}}$ is the measured temperature at the grid points.

The time domain in all cases is from 0 to 0.25, with a 0.01 increment. The increment of spatial coordinates is 0.1. The results of the estimation of the unknown conditions from the knowledge of the temperature at measurement points are examined.

Detailed descriptions for the example problems are shown in the following:

Example problems: the problem described from Eq. (1) to Eq. (6) (Fig. 1), with the given boundary conditions $\hat{f}_1(y, t)$ at $x = 1$ and $\hat{f}_2(x, t)$ at $y = 0$, are the polynomial forms with spatial and time variables:

$$\hat{f}_1(y, t) = 1 + 0.2t + 0.5y + 0.2y^2 + 0.3yt \quad 0 < t < 1 \tag{13}$$

$$\hat{f}_2(x, t) = 1.5 + 0.5t + 0.1x + 0.3x^2 + 0.4xt \quad 0 < x \leq 1 \tag{14}$$

The unknown boundary condition $f_1(y, t)$ at $x = 1$ and $f_2(x, t)$ at $y = 0$ is a polynomial form with spatial and time variables and can be approximated by the following forms:

$$f_1(y, t) = a_0 + a_1 t + a_2 y + a_3 y^2 + a_4 yt \qquad 0 < y < 1 \tag{15}$$

$$f_2(x, t) = b_0 + b_1 t + b_2 x + b_3 x^2 + b_4 xt \qquad 0 < x \leq 1 \tag{16}$$

In the first example, three measuring points are allocated at the following coordinates:

$$x = 0.7 \qquad y = 0.2$$

$$x = 0.8 \qquad y = 0.3$$

$$x = 0.8 \qquad y = 0.2$$

In the second example, five thermocouples are allocated at the following coordinates:

$$x = 0.7 \qquad y = 0.2$$

$$x = 0.8 \qquad y = 0.3$$

$$x = 0.8 \qquad y = 0.2$$

$$x = 0.6 \qquad y = 0.2$$

$$x = 0.8 \qquad y = 0.4$$

In the third example, seven thermocouples are allocated at the following coordinates:

$$x = 0.7 \qquad y = 0.2$$

$$x = 0.8 \qquad y = 0.3$$

Table 1 Comparison of the exact solutions and the estimated results without measuring errors in examples 1, 2, and 3

Coefficients	Exact solution	Three-point measurement	Five-point measurement	Seven-point measurement
a_0	1.000000	0.9999999372	1.0000000003	0.9999999998
a_1	0.200000	0.1999977248	0.2000000001	0.2000000001
a_2	0.500000	0.5000001529	0.4999999978	0.5000000009
a_3	0.200000	0.1999999085	0.2000000027	0.1999999989
a_4	0.300000	0.3000046874	0.2999999997	0.2999999999
b_0	1.500000	1.5000000538	1.4999999996	1.5000000001
b_1	0.500000	0.5000017725	0.5000000001	0.4999999999
b_2	0.100000	0.0999998704	0.1000000021	0.0999999992
b_3	0.300000	0.3000000889	0.2999999975	0.3000000009
b_4	0.400000	0.3999963512	0.4000000000	0.4000000002

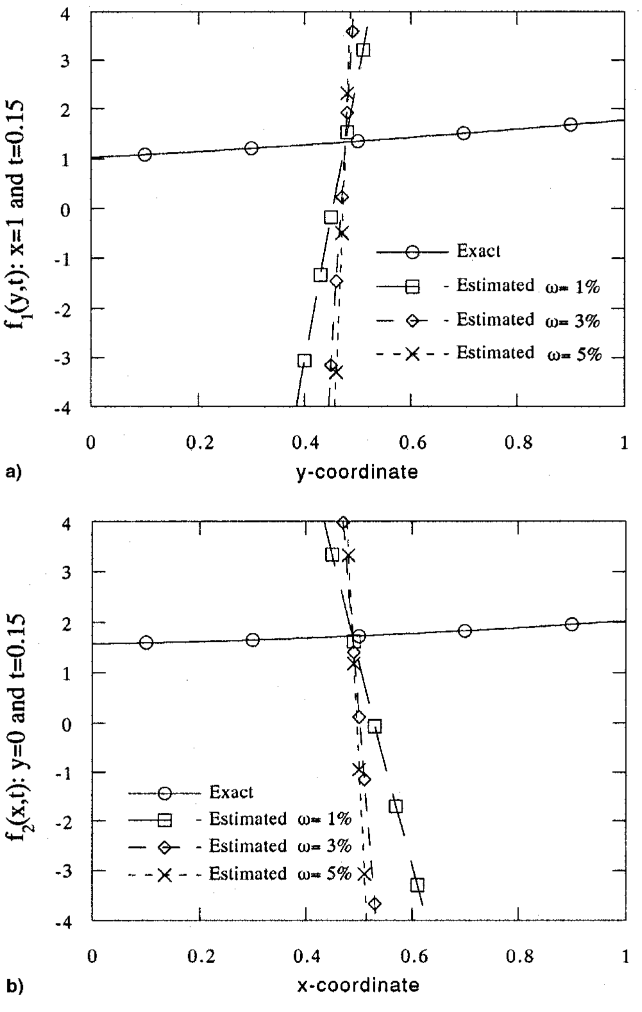

Fig. 3 Estimation of the boundary conditions a) $f_1(y, t)$ at $x = 1$ and b) $f_2(x, t)$ at $y = 0$ and $t = 0.15$ with three measuring points (measurement errors $\omega = 1, 3$, and 5%).

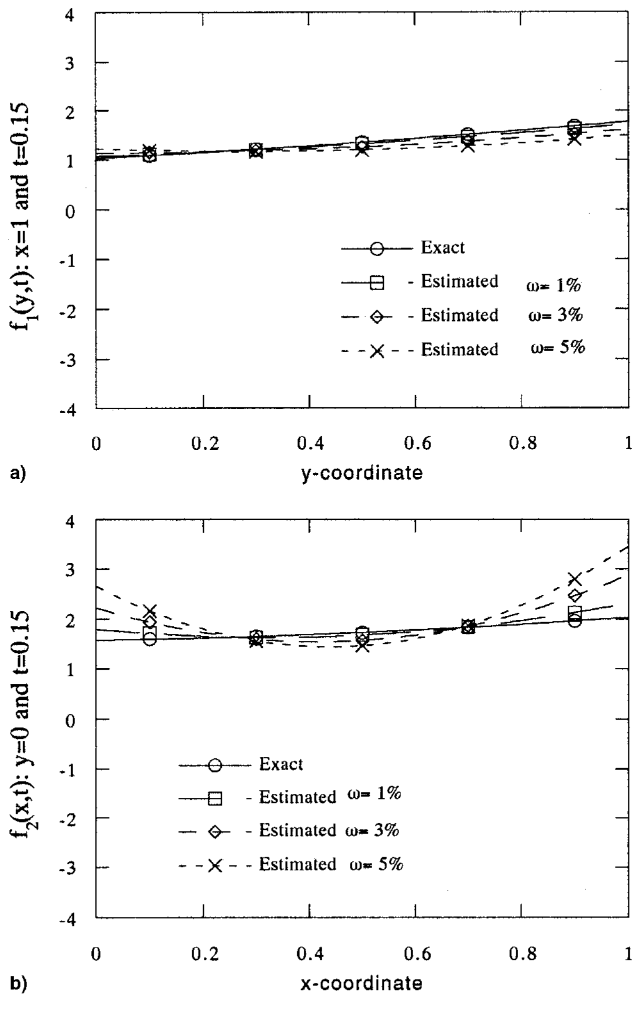

Fig. 4 Estimation of the boundary conditions a) $f_1(y, t)$ at $x = 1$ and b) $f_2(x, t)$ at $y = 0$ and $t = 0.15$ with five measuring points (measurement errors $\omega = 1, 3$, and 5%).

$$x = 0.8 \qquad y = 0.2$$

$$x = 0.6 \qquad y = 0.2$$

$$x = 0.8 \qquad y = 0.4$$

$$x = 0.8 \qquad y = 0.5$$

$$x = 0.5 \qquad y = 0.2$$

In examples 1, 2, and 3, the results without measurement errors are shown in Table 1. All three examples have good approximations when the measurements are error-free. The estimated result with measurement error (1, 3, and 5%) at time coordinates ($t = 0.15$) is shown in Figs. 3a and 3b for example 1, Figs. 4a and 4b for example 2, and Figs. 5a and 5b for example 3. In general, large errors make the estimated results diverge from the exact solution, and, the larger the further. For example, the results shown in Fig. 5b have the maximum deviation from their exact solutions when the error is equal to 5%. Moreover, it appears that the results in Figs. 3a–5a have a similar trend as that in Fig. 5b.

From the results for examples 1, 2, and 3, it is evident that there makes no difference in three individual measuring methods when there are no measurement errors. However, the difference becomes influential when the error of measurements

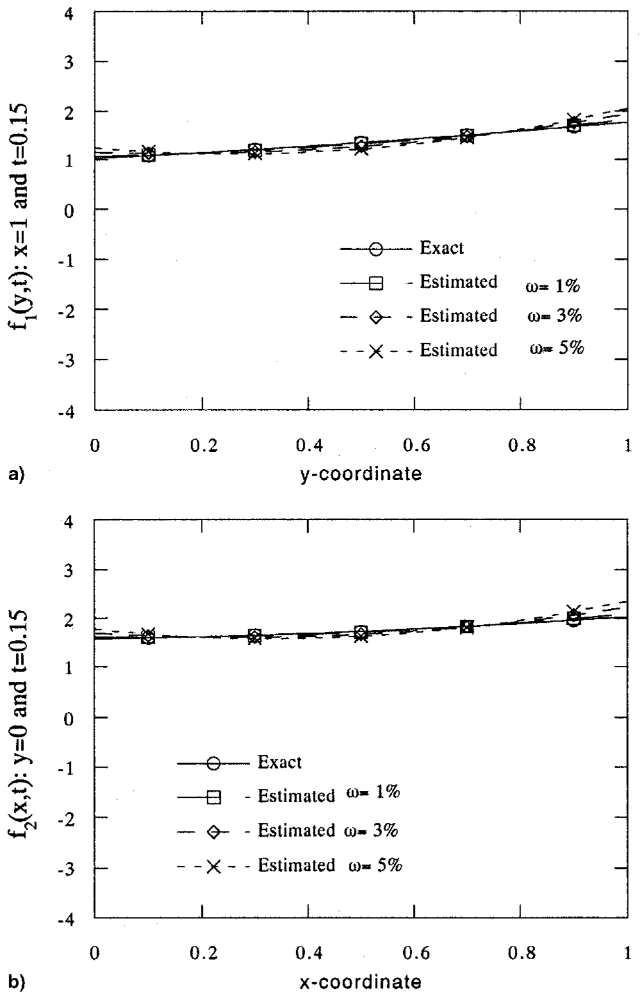

Fig. 5 Estimation of the boundary conditions a) $f_1(y, t)$ at $x = 1$ and b) $f_2(x, t)$ at $y = 0$ and $t = 0.15$ with seven measuring points (measurement errors $\omega = 1, 3,$ and 5%).

is considered. Generally speaking, the deviations in the three-point measuring method are unacceptable when the error appears (see Fig. 3). In the five-point measurement, even when the error is equal to 5%, the deviations are smaller than those in the three-point method at $\omega = 1\%$ (see Figs. 3 and 4). It is interesting to find that the deviations in the seven-point measuring method have a better approximation for the result when measurement errors are included.

Conclusions

The proposed symbolic approach has been introduced for solving the two-dimensional inverse conduction problems. An implicit alternating-direction finite difference method is employed to solve the symbolic temperature field when the boundary conditions are represented as symbols. A special feature about the proposed method is that the inverse computation is in a linear domain and the nonlinear optimization process used in the traditional approach can be eliminated. Three examples have been used to show the usage of the proposed method. From the results, it appears that the exact solution can be found through the proposed method when only three points are measured. Yet, this is under the condition that the measurement errors are zero. When measurement errors are included, it is suggested that the seven-point measuring method needs to be adopted for a better result in the example problems. This result can be refereed to allocate the number of measuring points in the two-dimensional inverse heat conduction problem in the future research. After all, the proposed method is applicable to the other kinds of inverse problems such as initial estimation and source strength estimation in the one- or multidimensional heat transfer problems.

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